

**MORE ADVENTURES IN SCALE GENERATION
ALONG A WILSONIAN PATH:
SOME PLAYFUL THOUGHTS**

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In this paper, we investigate Burt (1996) and the definition of the combination-product set. We formalize Burt's analysis and demonstrate how it can be used to calculate general tonal sets. As an example, we apply this formalization to a subset of the Niven numbers.

1.0 Introduction and Background. In his article "Adventures in Scale Generation Along a Wilsonian Path - Part 1," Burt (1996) introduces the reader to the concepts of a *combination-product set* and a *cross set*. I was intrigued by the use of this particular mathematical terminology, as well as its application to music theoretic concepts. I plead guilty to being an applied mathematician and a fledgling composer. However, whenever anyone uses mathematical terms in a music article, my interest piques.

In reading Burt's discussion, I realized that I was having a great deal of trouble trying to relate what was being said mathematically to what was actually being displayed in the examples. As I worked through the examples, I realized that there was a confusion between the use of the mathematical terms and their subsequent application. However, what was fascinating was that, in my attempt to clarify the discussion for myself, I realized that the concepts of Burt (1996) were far more complex and interesting than had first appeared. The purpose of this discussion is to clarify and expand upon the discussion of Burt (1996).

2.0 The Cross-Product. Okay, we have to do some mathematical discussion now. We begin our discussion with the following definition and example

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This article is dedicated to my former composition and conducting teacher William Albright (deceased), University of Michigan, Ann Arbor, who gave me the space and trust to spread my wings and begin to fly as a composer and conductor

Definition [1]. The cross product, denoted by the symbol \otimes of a set A with a set B , is the set $A \otimes B$ whose members are the ordered 2-tuples (a, b) where “ a ” is an element of the set A and “ b ” is an element of B .

This definition requires that, in order to construct the cross product set $A \otimes B$, we must have only elements of A as the first and only elements of B as the second members, respectively, of the 2-tuple (a, b) . Further, the element (a, b) is considered to be different from the element (b, a) if $a \neq b$. Let’s look at the following simple example.

Example [1a]. Suppose $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$, then the set $A \otimes B$ is given by

$$A \otimes B = \{(a_1, b_1); (a_1, b_2); (a_2, b_1); (a_2, b_2); (a_3, b_1); (a_3, b_2)\}.$$

Suppose that we now wish to discuss how many elements are in a set A . We call this number the “cardinality” of the set A and we denote this by $\text{card}(A)$. Suppose that the cardinality of the set A is n_A and the cardinality of the set B is n_B , then the following statement is true about the cardinality of the cross-product set $A \otimes A$

Lemma [1]. If $\text{card}(A) = n_A$ and $\text{card}(B) = n_B$, then

$$\text{card}(A \otimes B) = n_{A \otimes B} = n_A \cdot n_B.$$

Using Example [1a] above, we see that $\text{card}(A) = n_A = 3$, $\text{card}(B) = n_B = 2$, and it is clear that $\text{card}(A \otimes B) = n_{A \otimes B} = n_A \cdot n_B = 3 \cdot 2 = 6$. The proof of Lemma [1] is straightforward and is left to the interested reader as an exercise in mental gymnastics.

Let us now consider the cross-product of a set A with itself: $A \otimes A$ also denoted A^2 . By using Definition [1] above and applying it to the set A given in Example [1a], we can see that $A \otimes A$ is given by the following set

$$A \otimes A = \{(a_1, a_1); (a_1, a_2); (a_1, a_3); \\ (a_2, a_1); (a_2, a_2); (a_2, a_3); \\ (a_3, a_1); (a_3, a_2); (a_3, a_3)\}$$

Consider the following practical example.

Example [1b]. Suppose $A = \{2, 3, 5, 7\}$, then the set $A \otimes A$ is given by

$$A \otimes A = \{(2, 2); (2, 3); (2, 5); (2, 7); \\ (3, 2); (3, 3); (3, 5); (3, 7); \\ (5, 2); (5, 3); (5, 5); (5, 7); \\ (7, 2); (7, 3); (7, 5); (7, 7)\}.$$

From Lemma [1] above, we can see that the number of elements in the set $A \otimes A$ is just $\text{card}(A^2) = n_A \cdot n_A = 4 \cdot 4 = 16$, as is illustrated in the example above.

Up to this point, we have made use of the standard definition of the cross product of two sets. What we would like to do now is to give a formal definition of the “combination-product set” and the “cross set” as discussed in Burt (1996). To do this we will need to introduce another concept called a “relation.”

3.0 The Relation. We will denote a relation by the symbol R and we will say that two items a and b are related to each other by a relation R if aRb . Now, don't run screaming madly from the room. We will clarify this in just a bit. There are many different kinds of relations. In fact, we work with relations as a natural part of our lives. Most often, we don't even realize that we are using them. For example, the relation a is greater than b is denoted $a > b$. This is a relation where R is the $>$ relation. Another example of a relation is the equivalence of notes in a 12-tone scale under the operation of $\text{mod } 12$. That is, a note " a " is related to a note " b " in a 12-tone row if $a \equiv b \pmod{12}$. This is a particular kind of relational operation. It not only defines two things as being related to each other, but it also defines them as being **equivalent** under that relation. So, note 48 and note 24 are equivalent under the 12-tone $\text{mod } 12$ operation. Consequently, given a 12 tone row, all notes that satisfy the above relationship are said to be "**equivalent**" under the operation of $\text{mod } 12$. Suppose we were given the rather odd 12-tone row that began with the following sequence 1, 5, 13, 25, 17, ... The sets $\{1, 13, 25\}$ and $\{5, 17\}$ represent sets of notes that are all seen as equivalent to each other under the $\text{mod } 12$ operation. Notice that no element in one of these sets is equivalent to an element in the other set. That is, the sets $\{1, 13, 25\}$ and $\{5, 17\}$ are disjoint (share no elements in common). We call sets like this "**equivalence classes**" under the relation R . We denote each of these sets respectively as $[1]_R$ and $[5]_R$ to indicate all of the notes equivalent to 1 and to 5 respectively under the relation R . The concept of "**equivalence class**" is going to be very important to us in a moment.

There is no reason to suppose that we cannot define relations on 2-tuples and further, define relations in which 2-tuples are equivalent. Let us look further at this.

4.0 The Equivalence Class. Let us now define the concept of an equivalence class.

Definition [2]. Suppose that we define a relation R on a set $A \otimes B$ as follows. For any two 2-tuples (a_1, b_1) and (a_2, b_2) in the cross product set $A \otimes B$, then $(a_1, b_1)R(a_2, b_2)$ if the conditions $a_1 = b_2$ and $a_2 = b_1$ are simultaneously true. If the two 2-tuples under examination satisfy this set of two conditions, we say that (a_1, b_1) and (a_2, b_2) are "**equivalent**" under the action of R .

We can visualize this using the following Figure [1].

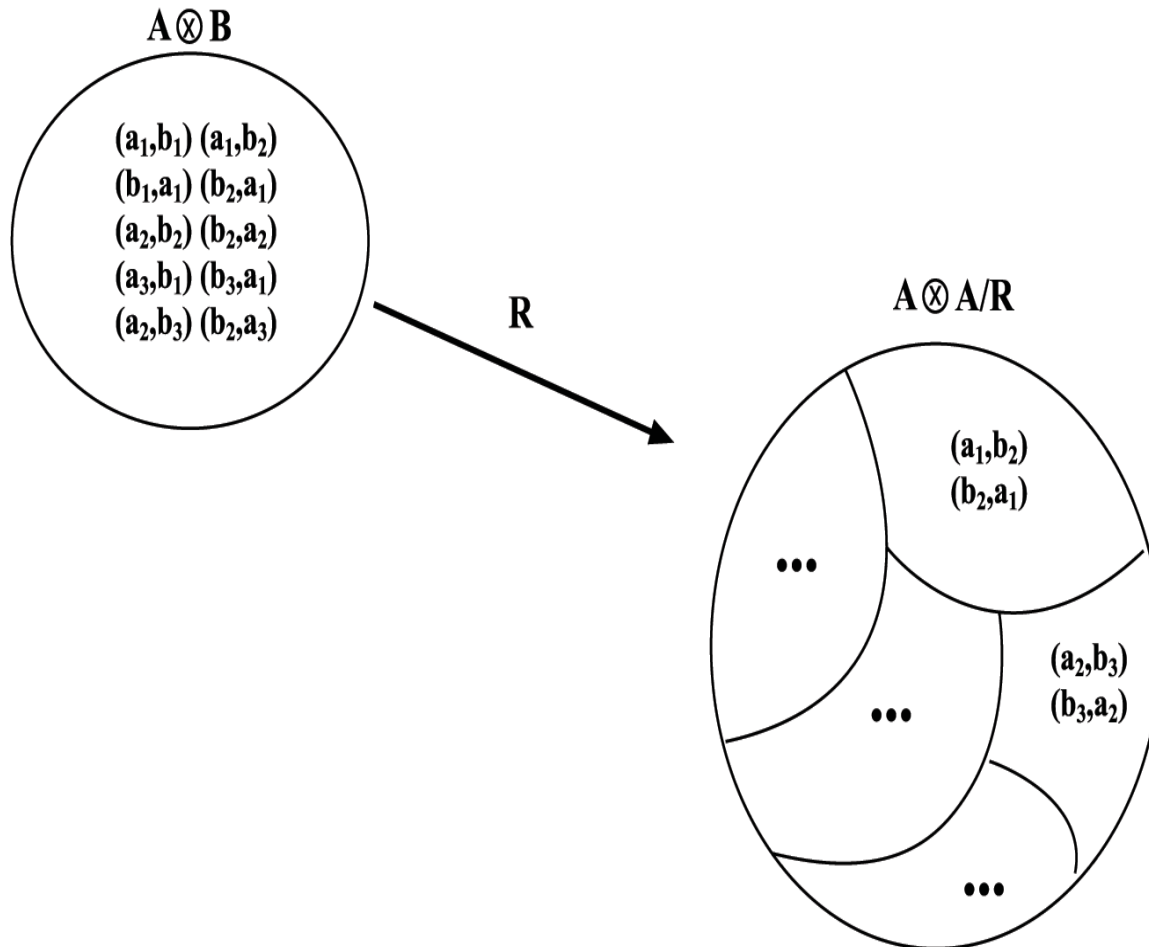


Figure [1]: An illustration of the effect of the equivalence mapping R on the set $A \otimes A$.

Let's look at an example to illustrate these concepts.

Example [2]. Suppose we let A and B be the following sets $A = \{2, 3, 5\}$ and $B = \{2, 3\}$. From our previous discussion, we can show that the cross product set $A \otimes B$ is given by

$$A \otimes B = \{(2, 2); (2, 3); (3, 2); (3, 3); (5, 2); (5, 3)\}.$$

Following Definition [2] for the relation R between pairs of 2-tuples, we see that the following sets of 2-tuple pairs in $A \otimes B$ would be seen as being equivalent under the relation R

$$\{(2, 2)\}; \{(2, 3), (3, 2)\}; \{(3, 3)\}; \{(5, 2)\}; \{(5, 3)\}$$

where each of these sets is an equivalence class under the operation R . We denote those sets by these equivalence relations.

$$[(2, 2)]_R; [(2, 3)]_R; [(3, 3)]_R; [(5, 2)]_R \text{ and } [(5, 3)]_R$$

noting that only the equivalence class $[(2, 3)]_R$ has more than one member.

What the relation R does to the cross product set $A \otimes B$ is to break $A \otimes B$ up into what are called “**equivalence classes**” under the action of the relation R . This is mathematically symbolized as follows

$$A \otimes B \xrightarrow{R} A \otimes B/R$$

where an element of $A \otimes B/R$ is denoted $[(a, b)]_R$ and is a set containing all of the 2-tuples that are equivalent to (a, b) under the action of R . Don’t panic, this is a fancy mathematical way of saying that all of the tuples in a given equivalence class look the same when we view them through the R microscope (look at Figure [1] again). Let’s look at another example.

Example [3]. Let $A = \{a_1, a_2, a_3\}$, and let R be as in Definition [2], then $A \otimes A$ is given by the following set

$$A \otimes A = \{(a_1, a_1); (a_1, a_2); (a_1, a_3); (a_2, a_1); (a_2, a_2); (a_2, a_3); (a_3, a_1); (a_3, a_2); (a_3, a_3)\}$$

and $A \otimes A/R$ is given by

$$A \otimes A/R = \{[(a_1, a_1)]_R; [a_2, a_2]_R; [a_3, a_3]_R; [a_1, a_2]_R; [a_2, a_3]_R; [a_1, a_3]_R\}$$

Observe that $\text{card}(A \otimes A/R) = 6 = 3 \cdot 2 \cdot 1 = 3!$ (3 factorial). We illustrate what is happening in Figure [2].

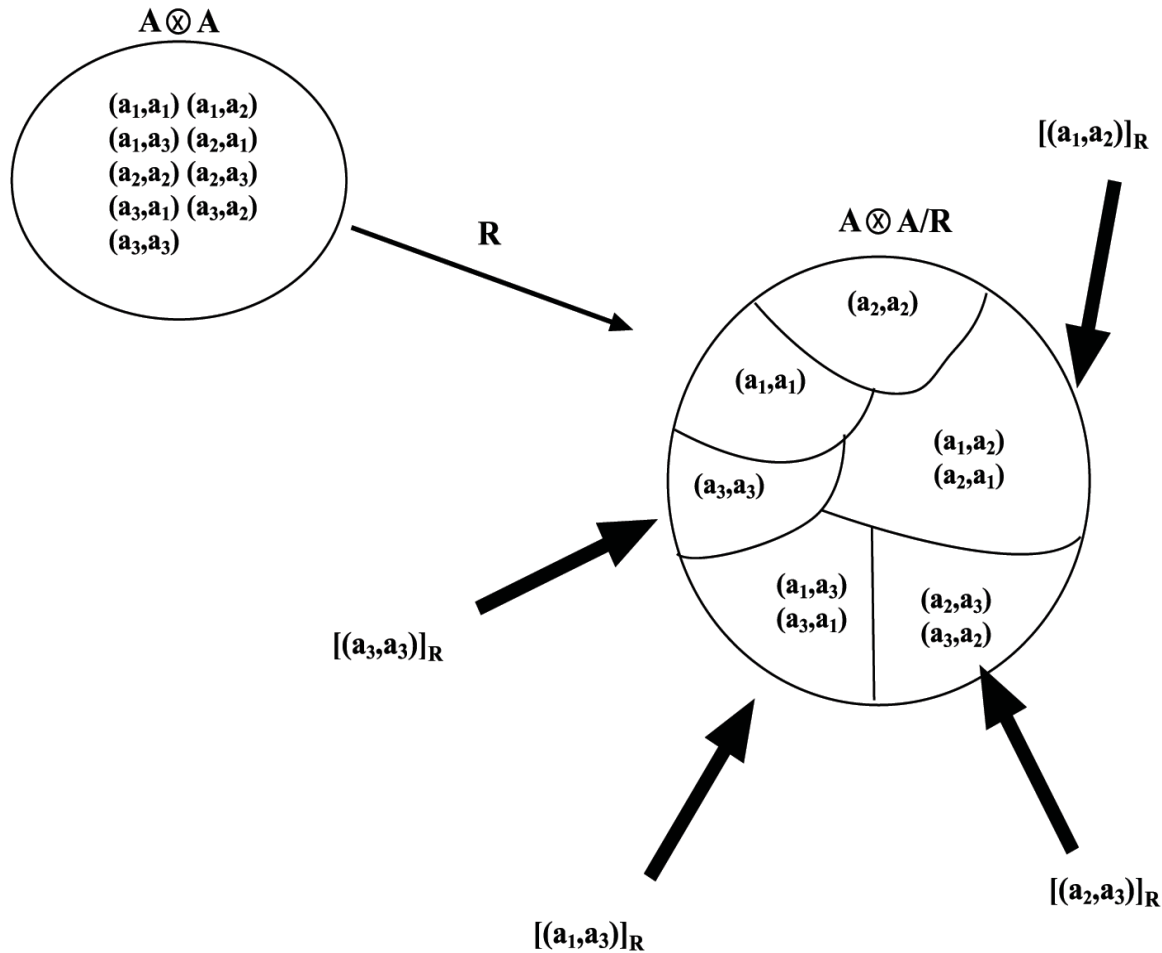


Figure [2]: An illustration of the effect of U on $A \otimes A/R$.

Stay with me here. We need one more definition and then we will be ready to look at Burt (1996) again.

Definition [3]. Let U be the “useless” operator defined as follows. We say that an equivalence class $[(a, b)]_R$ in $A \otimes B/R$ is useless if, whenever $a = b$, $U([(a, b)]_R) = \{ \cdot \}$ where \cdot is the “null” or “empty” set. That is, when $a = b$ the equivalence class $[(a, b)]_R$ gets sent (or mapped) into the empty set \cdot .

Okay, I know, you’re thinking what does this have to do with anything?! One more picture and a bit of discussion and we are home free and we can talk music again.

Since U is an operation on the elements $[(a, b)]_R$ which are members of $A \otimes B/R$, it acts on the set $A \otimes B/R$, and carries it into another set $A \otimes B/R/U$. In fact, it acts as follows

$$A \otimes B/R \xrightarrow{U} A \otimes B/R/U$$

where $A \otimes B/R/U$ is the set of elements (a, b) that are first equivalent under R and which are then subsequently equivalent under U . This may be a bit difficult to see in the abstract so let's construct an example that might help us here.

Example [4]. Because we cannot illustrate the upcoming discussion with colored typeface, we will use a color tag in the following way. Suppose that we use our original two sets A and B as given in Example [2] above; $A = \{2, 3, 5\}$ and $B = \{2, 3\}$. Further, imagine that we could typeset these sets using colors so that we could typeset a red 2 and a blue 2, if we wanted to do so. Additionally, suppose that we could do this with each of the numbers in each of the two sets A and B above. Define the following new sets A and B where

$$A = \{ \boxed{2|red}, \boxed{3|red}, \boxed{5|blue} \}$$

and

$$B = \{ \boxed{2|blue}, \boxed{3|red} \}$$

The cross product set $A \otimes B$ is given(Definition [1]) by

$$A \otimes B = \{ \left(\boxed{2|red}, \boxed{2|blue} \right); \left(\boxed{2|red}, \boxed{3|red} \right); \\ \left(\boxed{3|red}, \boxed{2|blue} \right); \left(\boxed{3|red}, \boxed{3|red} \right); \\ \left(\boxed{5|blue}, \boxed{2|blue} \right); \left(\boxed{5|blue}, \boxed{3|red} \right) \}$$

For the moment, ignore the imaginary coloring of the typeface and construct the equivalence classes of $A \otimes B$ under R (Definition [2]). These classes are given by the following

$$A \otimes B/R = \{ \left(\boxed{2|red}, \boxed{2|blue} \right) \}, \\ \{ \left(\boxed{2|red}, \boxed{3|red} \right); \left(\boxed{3|red}, \boxed{2|blue} \right) \}, \\ \{ \left(\boxed{3|red}, \boxed{3|red} \right) \}, \\ \{ \left(\boxed{5|blue}, \boxed{2|blue} \right) \}, \\ \{ \left(\boxed{5|blue}, \boxed{3|red} \right) \}$$

Compare this to the result given in Example [2] where there were no color tags. Observe that the equivalence set $A \otimes B/R$ actually contains different elements when we retain the coloring of the typeface. That is, it is only when we “forget” the coloring that the two elements in this set are seen as equivalent.

5.0 Burt's Cross-Product Set. Now, let's return to Burt (1996) and the examples provided therein and see how the previous discussion can be used to elaborate on what is actually being constructed. Burt's example is as follows.

Example [5]. Let $A = \{2, 3, 5, 7, 11\}$. Clearly A contains 5 elements. Hence, $\text{card}(A) = 5$. The cross product set $A \otimes A$ has $5 \cdot 5 = 25$ elements ($\text{card}(A \otimes A) = 25$) and is constructed in the obvious way. That is, we construct all possible ordered pairs of elements of A with elements of A . $A \otimes A$ is

$$\begin{aligned} A \otimes A = \{ & (2, 2); (2, 3); (2, 5); (2, 7); (2, 11); \\ & (3, 2); (3, 3); (3, 5); (3, 7); (3, 11); \\ & (5, 2); (5, 3); (5, 5); (5, 7); (5, 11); \\ & (7, 2); (7, 3); (7, 5); (7, 7); (7, 11); \\ & (11, 2); (11, 3); (11, 5); (11, 7); (11, 11)\} \end{aligned}$$

From Definition [2] for R , we see that the action of R on $A \otimes A$ is to create the following new set $A \otimes A/R$ of equivalence classes, which is given as follows

$$\begin{aligned} A \otimes A/R = \{ & [(2, 2)]_R; [(2, 3)]_R; [(2, 5)]_R; [(2, 7)]_R; [(2, 11)]_R; \\ & [(3, 3)]_R; [(3, 5)]_R; [(3, 7)]_R; [(3, 11)]_R; \\ & [(5, 5)]_R; [(5, 7)]_R; [(5, 11)]_R; \\ & [(7, 7)]_R; [(7, 11)]_R; \\ & [(11, 11)]_R \} \end{aligned}$$

where each of the equivalence classes $[(\cdot, \cdot)]$ is actually a set of 2-tuples containing the following elements

$$\begin{aligned} [(2, 2)]_R &= \{(2, 2)\} \\ [(2, 3)]_R &= \{(2, 3); (3, 2)\} \\ [(2, 5)]_R &= \{(2, 5); (5, 2)\} \\ [(2, 7)]_R &= \{(2, 7); (7, 2)\} \\ [(2, 11)]_R &= \{(2, 11); (11, 2)\} \\ [(3, 3)]_R &= \{(3, 3)\} \\ [(3, 5)]_R &= \{(3, 5); (5, 3)\} \\ [(3, 7)]_R &= \{(3, 7); (7, 3)\} \\ [(3, 11)]_R &= \{(3, 11); (11, 3)\} \\ [(5, 5)]_R &= \{(5, 5)\} \\ [(5, 7)]_R &= \{(5, 7); (7, 5)\} \\ [(5, 11)]_R &= \{(5, 11); (11, 5)\} \\ [(7, 7)]_R &= \{(7, 7)\} \\ [(7, 11)]_R &= \{(7, 11); (11, 7)\} \\ [(11, 11)]_R &= \{(11, 11)\} \end{aligned}$$

By simply counting the number of equivalence classes $[(\cdot, \cdot)]$, we observe that $\text{card}(A \otimes A/R) = 15$. If we now act U on the set $A \otimes A/R$, we then obtain the new equivalence class set $A \otimes A/R/U$

which is given as follows

$$A \otimes A/R/U = \{[(2, 3)]_R; [(2, 5)]_R; [(2, 7)]_R; [(2, 11)]_R; \\ [(3, 5)]_R; [(3, 7)]_R; [(3, 11)]_R; \\ [(5, 7)]_R; [(5, 11)]_R; \\ [(7, 11)]_R\}$$

and is constructed by simply removing all of the equivalence class elements of the form $[(x, x)]$ in the set $A \otimes A/R$. There are 5 such sets, yielding the result that $\text{card}(A \otimes A/R/U) = 10$. Observe that the "useless operator" strips out the diagonal members of $A \otimes A/R$. Hence, the number of elements stripped away by U is just $\text{card}(A \otimes A/R) = n_A = 5$

This is exactly what Burt (1996) obtains on page 1, example 1 of his article. Hence, a **combination product set** (as defined in Burt (1996) and incorrectly using the cross product set terminology) is actually not a combination product set at all. Rather, it is a set of equivalent 2-tuples classified by the following relationship

$$A \otimes A \xrightarrow{R} A \otimes A/R \xrightarrow{U} \overbrace{A \otimes B/R}^{\text{CPS}} / U$$

This may be pictorially visualized by combining, Figures [1] and [2] in sequential order. It is the last portion of this diagram, the one with the overbrace, that is actually what Burt calls the **cross product set**.

6.0 How Many Are There. Having constructed all of these different sets, we might ask, as we increase the number of elements in each set, how does the size of each set increase?

Suppose we start with $A \otimes A$. If $\text{card}(A) = n_A$, we can show that the following statements are true.

Lemma [2]. *If $\text{card}(A) = n_A$, then $\text{card}(A \otimes A) = n_A^2$.*

Lemma [3]. *If $\text{card}(A) = n_A$, then*

- (a) $\text{card}(A \otimes A/R) = \frac{n_A(n_A+1)}{2}$ and,
 (b) $\text{card}(A \otimes A/R/U) = \frac{n_A(n_A-1)}{2}$.

Proof of Lemma [2] is straightforward and follows from a simple counting argument that goes as follows. Suppose my set of numbers A contains the elements $A = \{a_1, a_2, \dots, a_{n_A}\}$. To construct the set $A \otimes A$ we choose an arbitrary element a_j in A and we form all possible 2-tuples of the form $(a_j, a_k)_{k=1, \dots, n_A}$. Since there are exactly n_A possible values that a_k can take on, and since we can execute this procedure of forming all possible 2-tuples for each value of a_j $j = 1, \dots, n_A$, we will have exactly n_A sets containing n_A 2-tuples for a total of $n_A \cdot n_A = n_A^2$ 2-tuples.

The proof of Lemma [3a,b] is a bit more complex and goes as follows. Pick the element a_1 from the set A . Use all possible elements in A , including a_1 to construct all possible pairs of elements of the form $(a_1, a_1), (a_1, a_2), \dots, (a_1, a_{n_A})$. Since you have already used a_1 on the first pass of the construction, next pick an arbitrary element a_2 in A and use the remaining $(n_A - 1)$ elements to form the sets $(a_2, a_2), (a_2, a_3), \dots, (a_2, a_{n_A})$. Continue to repeat this process until

you have reached the last element $j = n_A$. Now count how many elements you have in the each phase of the construction process and you will see that

$$\text{card}(A \otimes A/R) = n_A + (n_A - 1) + (n_A - 2) + \dots + (n_A - (n_A - 1)) = \frac{n_A(n_A + 1)}{2}$$

The proof of Lemma [3b] is as follows.

$$\text{card}(A \otimes A/R/U) = \text{card}(A \otimes A/R) - \text{card}(A) = \frac{n_A(n_A + 1)}{2} - n_A = \frac{n_A(n_A - 1)}{2}$$

Hence, from Burt (1966)'s example, $n_A = 5$; $\text{card}(A \otimes A) = 25$; $\text{card}(A \otimes A/R) = 15$; and $\text{card}(A \otimes A/R/U) = 10$ which is exactly what we obtained by directly constructing the sets.

Table [1] Summary of the number of elements in the sets A , $A \otimes A$, $A \otimes A/R$, and $A \otimes A/R/U$ given that the number of elements in the set is denoted $\text{card}(A) = n_A$.

Set Name	Cardinality
A	n_A
$A \otimes A$	n_A^2
$A \otimes A/R$	$\frac{n_A(n_A+1)}{2}$
$A \otimes A/R/U$	$\frac{n_A(n_A-1)}{2}$

We plot the numbers, for each set in Table [1] above, in Figure [3] below.

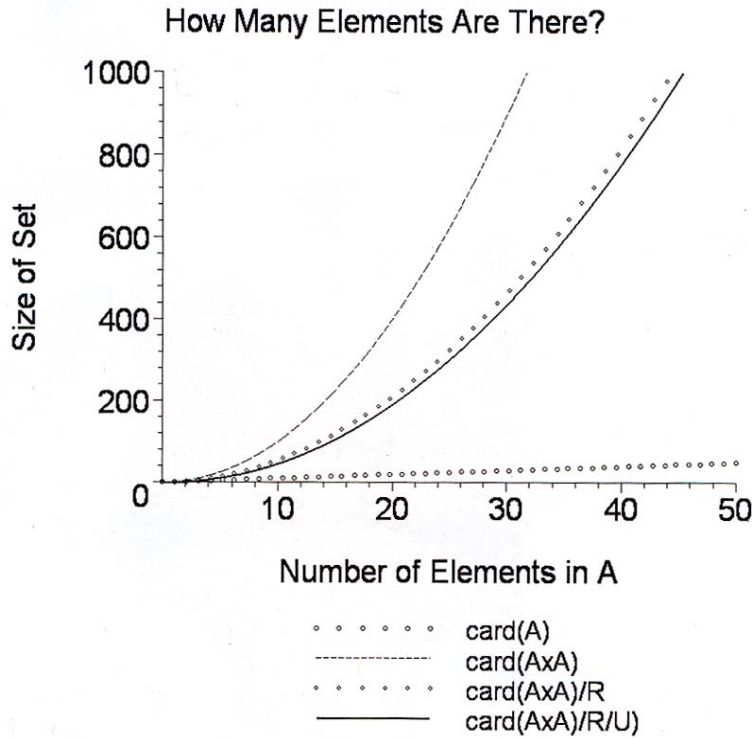


Figure [3]: A graph of $\text{card}(A \otimes A)$; $\text{card}(A \otimes A/R)$; and $\text{card}(A \otimes A/R/U)$ as a function of the number of elements in the set A given by n_A .

7.0 Music. Having constructed a specific CPS set $A \otimes A/R/U$, we can then take every pair (a_i, a_j) in the set and multiply the component elements a_i and a_j together to create a number which we shall denote as α_{ij} where $\alpha_{ij} = a_i \cdot a_j$. Observe that the elements α_{ij} form a triangular matrix A_R where, following Example [5], A_R would look like

$$A_R = \begin{pmatrix} 6 & 12 & 20 & 30 \\ \cdot & 8 & 15 & 24 \\ \cdot & \cdot & 10 & 18 \\ \cdot & \cdot & \cdot & 12 \end{pmatrix}$$

where \cdot means that there is no element in that position in the matrix and α_{ij} is the element in the i^{th} row and the j^{th} column of the matrix A_R .

We say a matrix A is 12-equitempered if we normalize each element α_{ij} in A such that we obtain a new matrix N_A whose elements η_{ij} satisfy that they are in the interval $[1, 2)$. It will be clearer if we actually look at an example at this point.

Example [6]. Suppose we consider the basis set $A = \{3, 5, 11, 17, 31\}$. The set $A \otimes A$ can be

represented in matrix form as follows

$$A \otimes A = \begin{pmatrix} (3,3) & (3,5) & (3,11) & (3,17) & (3,31) \\ (5,3) & (5,5) & (5,11) & (5,17) & (5,31) \\ (11,3) & (11,5) & (11,11) & (11,17) & (11,31) \\ (17,3) & (17,5) & (17,11) & (17,17) & (17,31) \\ (31,3) & (31,5) & (31,11) & (31,17) & (31,31) \end{pmatrix}$$

The set $A \otimes A/R$ is then given by the following matrix

$$A \otimes A/R = \begin{pmatrix} (3,3) & (3,5) & (3,11) & (3,17) & (3,31) \\ \cdot & (5,5) & (5,11) & (5,17) & (5,31) \\ \cdot & \cdot & (11,11) & (11,17) & (11,31) \\ \cdot & \cdot & \cdot & (17,17) & (17,31) \\ \cdot & \cdot & \cdot & \cdot & (31,31) \end{pmatrix}$$

and the set $A \otimes A/R/U$ is then given by the following matrix

$$A \otimes A/R/U = \begin{pmatrix} \cdot & (3,5) & (3,11) & (3,17) & (3,31) \\ \cdot & \cdot & (5,11) & (5,17) & (5,31) \\ \cdot & \cdot & \cdot & (11,17) & (11,31) \\ \cdot & \cdot & \cdot & \cdot & (17,31) \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

If we then take the non-empty elements a_{ij} in the matrix $A \otimes A/R/U$ above and multiply the component elements of each 2-tuple together, we obtain the dekany matrix D_A given by

$$D_A = \begin{pmatrix} 15 & 33 & 51 & 93 \\ \cdot & 55 & 85 & 155 \\ \cdot & \cdot & 187 & 341 \\ \cdot & \cdot & \cdot & 527 \end{pmatrix}$$

where $\alpha_{ij} \in D_A$ is the product of the two elements of the 2-tuple (a_i, a_j) . From the dekany matrix D_A we find the transformed dekany matrix D'_A . To do this we let β be the maximum value in the dekany matrix D_A . In the case of our ongoing example, $\beta = 527$. We also need 2β which is 1054. To create the transformed dekany matrix, we require that for each $\alpha_{ij} \in D_A$, find an integer value m_{ij} in $Z^+ \cup \{0\}$ such that (1) $2^{m_{ij}} \alpha_{ij} < 2\beta$ and (2) $2^{m_{ij}+1} \alpha_{ij} \geq 2\beta$. This yields the following transformed dekany matrix for our example

$$D'_A = \begin{pmatrix} 15 \cdot 2^5 & 33 \cdot 2^4 & 51 \cdot 2^4 & 93 \cdot 2^3 \\ \cdot & 55 \cdot 2^4 & 85 \cdot 2^3 & 155 \cdot 2^2 \\ \cdot & \cdot & 187 \cdot 2^2 & 341 \cdot 2^1 \\ \cdot & \cdot & \cdot & 527 \cdot 2^1 \end{pmatrix}$$

which yields the following matrix

$$D'_A = \begin{pmatrix} 480 & 528 & 816 & 744 \\ \cdot & 880 & 680 & 620 \\ \cdot & \cdot & 748 & 682 \\ \cdot & \cdot & \cdot & 527 \end{pmatrix}$$

Let κ be the smallest element in D'_A and divide each element of D'_A by κ . This yields the ℓ -normalized ($\ell = 12$) matrix N_A . For the example matrix $\kappa = 480$ and $N_A = \frac{1}{\kappa}D'_A$, which is given by the following matrix

$$N_A = \begin{pmatrix} 480/480 & 528/480 & 816/480 & 744/480 \\ \cdot & 880/480 & 680/480 & 620/480 \\ \cdot & \cdot & 748/480 & 682/480 \\ \cdot & \cdot & \cdot & 527/480 \end{pmatrix}$$

where $\eta_{ij} \in N_A$. Using the ratio to cents formula

$$\sigma(\text{cents}) = \left(\frac{1200}{\ln 2} \right) \ln \eta$$

we can convert the matrix N_A to a new cents matrix C_A whose elements σ_{ij} are given by the formula

$$\sigma_{ij} = \left(\frac{1200}{\ln 2} \right) \ln \eta_{ij}$$

For this particular example, our cents matrix is given by

$$C_A = \begin{pmatrix} 0 & 162 & 165 & 443 \\ \cdot & 603 & 608 & 759 \\ \cdot & \cdot & 768 & 919 \\ \cdot & \cdot & \cdot & 1040 \end{pmatrix}$$

From the cents matrix, we can then create a tonal set T_A which is just the ordered (ascending in values) tonal cents sequents derived from the elements of the cents matrix. For this particular example, $T_A = \{0, 162, 165, 443, 603, 608, 759, 768, 919, 1049\}$.

The usefulness of this algorithm is that it applies to any given set A whether or not the set contains duplicates or multiples. Let us look at an example in which this is the case.

Example [7]. Suppose we consider the basis set $A = \{2, 3, 4, 5, 6\}$. The set $A \otimes A$ can be represented in matrix form as follows

$$A \otimes A = \begin{pmatrix} (2, 2) & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 2) & (3, 3) & (3, 4) & (3, 5) & (3, 6) \\ (4, 2) & (4, 3) & (4, 4) & (4, 5) & (4, 6) \\ (5, 2) & (5, 3) & (5, 4) & (5, 5) & (5, 6) \\ (6, 2) & (6, 3) & (6, 4) & (6, 5) & (6, 6) \end{pmatrix}$$

The set $A \otimes A/R$ is then given by the following matrix

$$A \otimes A/R = \begin{pmatrix} (2, 2) & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ \cdot & (3, 3) & (3, 4) & (3, 5) & (3, 6) \\ \cdot & \cdot & (4, 4) & (4, 5) & (4, 6) \\ \cdot & \cdot & \cdot & (5, 5) & (5, 6) \\ \cdot & \cdot & \cdot & \cdot & (6, 6) \end{pmatrix}$$

and the set $A \otimes A/R/U$ is then given by the following matrix

$$A \otimes A/R/U = \begin{pmatrix} \cdot & (2,3) & (2,4) & (2,5) & (2,6) \\ \cdot & \cdot & (3,4) & (3,5) & (3,6) \\ \cdot & \cdot & \cdot & (4,5) & (4,6) \\ \cdot & \cdot & \cdot & \cdot & (5,6) \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

If we then take the non-empty elements a_{ij} in the matrix $A \otimes A/R/U$ above and multiply the component elements of each 2-tuple together, we obtain the dekany matrix D_A given by

$$D_A = \begin{pmatrix} 6 & 8 & 10 & 12 \\ \cdot & 12 & 15 & 18 \\ \cdot & \cdot & 20 & 24 \\ \cdot & \cdot & \cdot & 30 \end{pmatrix}$$

Following the previous discussion, $\beta = \max[\alpha_{ij}] = 30$ and $2\beta = 60$, where $\alpha_{ij} \in D_A$ is the product of the two elements of the 2-tuple (a_i, a_j) . From the dekany matrix D_A we find the transformed dekany matrix D'_A .

$$D'_A = \begin{pmatrix} 48 & 32 & 40 & 48 \\ \cdot & 48 & 30 & 36 \\ \cdot & \cdot & 40 & 48 \\ \cdot & \cdot & \cdot & 30 \end{pmatrix}$$

Let κ be the smallest element in D'_A and divide each element of D'_A by κ . This yields the ℓ -normalized ($\ell = 12$) matrix N_A . For the example matrix $\kappa = 30$ and $N_A = \frac{1}{\kappa}D'_A$, which is given by the following matrix

$$N_A = \begin{pmatrix} 48/30 & 32/30 & 40/30 & 48/30 \\ \cdot & 48/30 & 30/30 & 36/30 \\ \cdot & \cdot & 40/30 & 48/30 \\ \cdot & \cdot & \cdot & 30/30 \end{pmatrix}$$

Note that there are elements $\eta_{ij} \in D_A$ that are not unique. Keeping only the single unique elements and removing all of the duplicates, N_A becomes

$$N_A = \begin{pmatrix} 48/30 & 32/30 & 40/30 & \cdot \\ \cdot & \cdot & 30/30 & 36/30 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

The cents matrix C_A for the matrix N_A above is just

$$C_A = \begin{pmatrix} 813.7 & 111.7 & 498 & \cdot \\ \cdot & \cdot & 0 & 315.6 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

So that the tonal set T_A is given by $T_A = \{0, 112, 316, 498, 814\}$

8.0 The Tonality Diamond. Burt points out that it is possible to expand a CPS set to a new construction in which each element of the tonal set T_A is used as a numerator and a denominator with every other element of the set.

Suppose we are given a normalized matrix N_A with elements $\eta_{ij} \in N_A$ and suppose that all of the duplicates of N_A have been removed. Further, suppose we construct the matrix $A \otimes A/R/U$ from which N_A is derived. Order the elements $a_{ij} \in A \otimes A/R/U$ from smallest to largest, corresponding to the non-empty elements $\eta_{ij} \in N_A$. Let us call this set Ω_A with elements ω_i . Lastly, create the matrix $\Omega_A \otimes \Omega_A$ where the ij^{th} element is ϵ_{ij} defined as $\epsilon_{ij} = \frac{\omega_j}{\omega_i}$. Let us clarify with an example.

Example [8]. Let $A = \{2, 3, 5, 6, 11\}$. Then $A \otimes A/R/U$ is just

$$A \otimes A/R/U = \begin{pmatrix} \cdot & (2,3) & (2,5) & (2,6) & (2,11) \\ \cdot & \cdot & (3,5) & (3,6) & (3,11) \\ \cdot & \cdot & \cdot & (5,6) & (5,11) \\ \cdot & \cdot & \cdot & \cdot & (6,11) \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and

$$D_A = \begin{pmatrix} 6 & 10 & 12 & 22 \\ \cdot & 15 & 18 & 33 \\ \cdot & \cdot & 30 & 55 \\ \cdot & \cdot & \cdot & 66 \end{pmatrix}$$

From this it follows that D'_A is given by

$$D'_A = \begin{pmatrix} 96 & 80 & 96 & 88 \\ \cdot & 120 & 72 & 66 \\ \cdot & \cdot & 120 & 110 \\ \cdot & \cdot & \cdot & 66 \end{pmatrix}$$

The tonal matrix N_A is given by

$$N_A = \begin{pmatrix} 96/66 & 80/66 & 96/66 & 88/66 \\ \cdot & 120/66 & 72/66 & 66/66 \\ \cdot & \cdot & 120/66 & 110/66 \\ \cdot & \cdot & \cdot & 66/66 \end{pmatrix} \text{ which simplifies to } \begin{pmatrix} 16/11 & 40/33 & \cdot & 44/33 \\ \cdot & 60/33 & 12/11 & 1 \\ \cdot & \cdot & \cdot & 55/33 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

or

$$N_A = \begin{pmatrix} 1.4545 & 1.2121 & \cdot & 1.3333 \\ \cdot & 1.8182 & 1.0909 & 1 \\ \cdot & \cdot & \cdot & 1.6667 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

From which we compute the cents matrix C_A

$$C_A = \begin{pmatrix} 648.68 & 333.04 & \cdot & 498.04 \\ \cdot & 1035.00 & 150.64 & 0 \\ \cdot & \cdot & \cdot & 884.36 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

If we now order the elements of N_A from smallest to largest, and pick the corresponding elements of $A \otimes A/R/U$ we obtain the set $\Omega_A = \{(3, 11), (3, 6), (2, 5), (2, 11), (2, 3), (5, 11), (3, 5)\}$. The diamond $\Omega_A \otimes \Omega_A$ would then be given by

$$\Omega_A \otimes \Omega_A = \begin{pmatrix} \frac{(3,11)}{(3,11)} & \frac{(3,6)}{(3,11)} & \frac{(2,5)}{(3,11)} & \frac{(2,11)}{(3,11)} & \frac{(2,3)}{(3,11)} & \frac{(5,11)}{(3,11)} & \frac{(3,5)}{(3,11)} \\ \frac{(3,11)}{(3,6)} & \frac{(3,6)}{(3,6)} & \frac{(2,5)}{(3,6)} & \frac{(2,11)}{(3,6)} & \frac{(2,3)}{(3,6)} & \frac{(5,11)}{(3,6)} & \frac{(3,5)}{(3,6)} \\ \frac{(3,11)}{(2,5)} & \frac{(3,6)}{(2,5)} & \frac{(2,5)}{(2,5)} & \frac{(2,11)}{(2,5)} & \frac{(2,3)}{(2,5)} & \frac{(5,11)}{(2,5)} & \frac{(3,5)}{(2,5)} \\ \frac{(3,11)}{(2,11)} & \frac{(3,6)}{(2,11)} & \frac{(2,5)}{(2,11)} & \frac{(2,11)}{(2,11)} & \frac{(2,3)}{(2,11)} & \frac{(5,11)}{(2,11)} & \frac{(3,5)}{(2,11)} \\ \frac{(3,11)}{(2,3)} & \frac{(3,6)}{(2,3)} & \frac{(2,5)}{(2,3)} & \frac{(2,11)}{(2,3)} & \frac{(2,3)}{(2,3)} & \frac{(5,11)}{(2,3)} & \frac{(3,5)}{(2,3)} \\ \frac{(3,11)}{(5,11)} & \frac{(3,6)}{(5,11)} & \frac{(2,5)}{(5,11)} & \frac{(2,11)}{(5,11)} & \frac{(2,3)}{(5,11)} & \frac{(5,11)}{(5,11)} & \frac{(3,5)}{(5,11)} \\ \frac{(3,11)}{(3,5)} & \frac{(3,6)}{(3,5)} & \frac{(2,5)}{(3,5)} & \frac{(2,11)}{(3,5)} & \frac{(2,3)}{(3,5)} & \frac{(5,11)}{(3,5)} & \frac{(3,5)}{(3,5)} \end{pmatrix}$$

Removing the diagonal elements from the previous matrix $\Omega_A \otimes \Omega_A/U$ yields the following matrix

$$(\Omega_A \otimes \Omega_A)' = \begin{pmatrix} \cdot & \frac{(3,6)}{(3,11)} & \frac{(2,5)}{(3,11)} & \frac{(2,11)}{(3,11)} & \frac{(2,3)}{(3,11)} & \frac{(5,11)}{(3,11)} & \frac{(3,5)}{(3,11)} \\ \frac{(3,11)}{(3,6)} & \cdot & \frac{(2,5)}{(3,6)} & \frac{(2,11)}{(3,6)} & \frac{(2,3)}{(3,6)} & \frac{(5,11)}{(3,6)} & \frac{(3,5)}{(3,6)} \\ \frac{(3,11)}{(2,5)} & \frac{(3,6)}{(2,5)} & \cdot & \frac{(2,11)}{(2,5)} & \frac{(2,3)}{(2,5)} & \frac{(5,11)}{(2,5)} & \frac{(3,5)}{(2,5)} \\ \frac{(3,11)}{(2,11)} & \frac{(3,6)}{(2,11)} & \frac{(2,5)}{(2,11)} & \cdot & \frac{(2,3)}{(2,11)} & \frac{(5,11)}{(2,11)} & \frac{(3,5)}{(2,11)} \\ \frac{(3,11)}{(2,3)} & \frac{(3,6)}{(2,3)} & \frac{(2,5)}{(2,3)} & \frac{(2,11)}{(2,3)} & \cdot & \frac{(5,11)}{(2,3)} & \frac{(3,5)}{(2,3)} \\ \frac{(3,11)}{(5,11)} & \frac{(3,6)}{(5,11)} & \frac{(2,5)}{(5,11)} & \frac{(2,11)}{(5,11)} & \frac{(2,3)}{(5,11)} & \cdot & \frac{(3,5)}{(5,11)} \\ \frac{(3,11)}{(3,5)} & \frac{(3,6)}{(3,5)} & \frac{(2,5)}{(3,5)} & \frac{(2,11)}{(3,5)} & \frac{(2,3)}{(3,5)} & \frac{(5,11)}{(3,5)} & \cdot \end{pmatrix}$$

Cancelling similar terms in the numerators and denominators, multiplying any elements that don't cancel, and then expressing the result as a fraction, yields the new matrix

$$(\Omega_A \otimes \Omega_A)'' = \begin{pmatrix} \cdot & \frac{6}{11} & \frac{10}{33} & \frac{2}{3} & \frac{2}{11} & \frac{5}{3} & \frac{5}{11} \\ \frac{11}{6} & \cdot & \frac{10}{18} & \frac{22}{18} & \frac{2}{6} & \frac{55}{18} & \frac{5}{6} \\ \frac{33}{10} & \frac{18}{10} & \cdot & \frac{11}{5} & \frac{3}{5} & \frac{5}{2} & \frac{15}{2} \\ \frac{3}{2} & \frac{18}{22} & \frac{5}{11} & \cdot & \frac{3}{11} & \frac{5}{2} & \frac{15}{22} \\ \frac{11}{2} & \frac{6}{2} & \frac{5}{3} & \frac{11}{3} & \cdot & \frac{55}{6} & \frac{5}{2} \\ \frac{3}{5} & \frac{18}{55} & \frac{2}{11} & \frac{2}{5} & \frac{6}{55} & \cdot & \frac{3}{11} \\ \frac{11}{5} & \frac{6}{5} & \frac{2}{3} & \frac{22}{15} & \frac{2}{5} & \frac{11}{3} & \cdot \end{pmatrix}$$

From $(\Omega_A \otimes \Omega_A)''$, we compute the ℓ -normalized matrix N_A . While doing this, we remove all duplicates. N_A given by

$$N_A = \begin{pmatrix} \cdot & \frac{12}{11} & \frac{40}{33} & \frac{4}{3} & \frac{16}{11} & \frac{5}{3} & \frac{20}{11} \\ \frac{11}{6} & \cdot & \frac{10}{9} & \frac{11}{9} & \cdot & \frac{55}{36} & \cdot \\ \frac{33}{20} & \frac{9}{5} & \cdot & \frac{11}{10} & \frac{6}{5} & \frac{11}{8} & \frac{3}{2} \\ \cdot & \frac{18}{11} & \cdot & \cdot & \cdot & \frac{5}{4} & \frac{15}{11} \\ \frac{11}{8} & \cdot & \cdot & \cdot & \cdot & \frac{55}{48} & \cdot \\ \cdot & \frac{72}{55} & \frac{16}{11} & \frac{8}{5} & \frac{96}{55} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{22}{15} & \cdot & \cdot & \cdot \end{pmatrix}$$

Using the ratio to cents formula given earlier, we can compute that cents matrix C_A which is given by

$$C_A = \begin{pmatrix} \cdot & 150.64 & 330.04 & 498.04 & 648.68 & 884.36 & 1035.00 \\ 1049.36 & \cdot & 182.40 & 347.41 & \cdot & 733.72 & \cdot \\ 866.96 & 1017.60 & \cdot & 165.00 & 315.64 & 551.32 & 702.00 \\ \cdot & 852.60 & \cdot & \cdot & \cdot & 386.31 & 536.95 \\ 551.32 & \cdot & \cdot & \cdot & \cdot & 235.68 & \cdot \\ \cdot & 466.28 & 648.68 & 813.69 & 964.32 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 663.05 & \cdot & \cdot & \cdot \end{pmatrix}$$

Ordering the elements of this set provides us with our tonality diamond tonal scale T_A

We close this discussion with an application of this theoretical construct to a set A defined by a subset of the Niven numbers. Niven numbers (or Harshad numbers), are positive integers that are divisible by the sum of their digits. The first twenty Niven numbers are $\{1, 2, 3, 5, 7, 8, 9, 10, 12, 18, 21, 24, 27, 30, 36, 40, 42, 45, 48, 50, \dots\}$. Choosing the set A to be an arbitrary subset of this set, we define $A = \{3, 7, 10, 21, 27, 45\}$. Without demonstration of all of the intermediate calculations, the tonal matrix for this set is N_A given by

$$N_A = \begin{pmatrix} \frac{112}{105} & \frac{32}{21} & \frac{168}{105} & \frac{324}{315} & \frac{12}{7} \\ \cdot & \frac{112}{63} & \frac{588}{315} & \frac{126}{105} & \frac{1}{7} \\ \cdot & \cdot & \frac{4}{3} & \cdot & \frac{10}{7} \\ \cdot & \cdot & \cdot & \frac{567}{315} & \frac{3}{2} \\ \cdot & \cdot & \cdot & \cdot & \frac{27}{14} \end{pmatrix}$$

which converts to a cents matrix C_A of the form

$$C_A = \begin{pmatrix} 111.73 & 729.22 & 813.69 & 54.28 & 933.13 \\ \cdot & 996.09 & 1080.56 & 315.64 & 0 \\ \cdot & \cdot & 498.04 & \cdot & 617.49 \\ \cdot & \cdot & \cdot & 999.18 & 701.96 \\ \cdot & \cdot & \cdot & \cdot & 1137.04 \end{pmatrix}$$

By ordering the elements of this matrix from lowest to highest, we obtain our tonal scale T_A .

$$T_A = \{0, 54.28, 111.73, 315.64, \\ 498.04, 617.49, 701.96, 719.22, \\ 813.69, 933.13, 996.09, 999.18, \\ 1080.56, 1137.04\}$$

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