A GENERAL SOLUTION OF THE TIME-INDEPENDENT FORWARD BOLTZMANN EQUATION FOR HIGH ENERGY ^{4}HE AND ITS SECONDARIES THROUGH BULK MATTER

Tarynn M. Witten*

Abstract. We construct a general solution to the time independent forward Boltzmann equation for high energy ^{4}He and its secondaries through bulk matter using the decomposition method.

1.0 The Forward Boltzmann Equation

1.1 Propagation of ^{4}He and its Secondaries Through Bulk Matter. The forward time independent Boltzmann equation for the propagation of ^{4}He and its secondaries through bulk matter is given, in the straight forward approximation [8] by

$$\left[\frac{\partial}{\partial x} - \nu_j \frac{\partial}{\partial E} S(E) + \sigma_j(E)\right] \phi_j(x, E) = \sum_k \int_0^\infty f_{jk}(E, E')\phi_k(x, E')dE' \tag{1}$$

where ν_j denotes the range-scaling parameter equal to Z_j^2/A_j, Z is the charge, and A is the mass number. In equation (1), $S(E)$ is the proton-stopping power, $\sigma(E)$ is the total cross section, $\phi_j(x, E)$ is the differential flux spectrum of type j ions, and $f_{jk}(E, E')$ is a differential energy cross section for redistribution of particle type and energy.

Using the following definitions

$$r = \int_0^E \frac{dE'}{S(E')} \tag{2}$$

$$\Psi_j(x, r) = S(E)\phi_j(x, E) \tag{3}$$

$$\overline{f}_{jk}(r, r') = S(E)f_{jk}(E, E') \tag{4}$$

Key words and phrases. Forward Boltzmann Equation, Adomian Polynomials, Decomposition Method, Space Radiation, Partial Differential Equations.
*Center for the Study of Biological Complexity, VCU Life Sciences Building, Suite 111 1000 West Cary Street, PO Box 842030, Richmond, VA 23284-2030
Email: twitten@vcu.edu

Running Head — Boltzmann Equation - January 2003

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Wilson[9-12] and Cucinotta[3-8] demonstrate that equation (1) may be re-expressed as follows

$$\left[\frac{\partial}{\partial x} - \nu_j \frac{\partial}{\partial r} + \sigma_j(r)\right] \Psi_j(x,r) = \sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_k(x,r')dr' \tag{5a}$$

Wilson et al. (refs 5 and 6) have solved equation (5a), yielding

$$\begin{aligned}\Psi_j(x,r) &= \exp\left[-\zeta_j(r,x)\right]\Psi_j(0, r+\nu_j x) \\ &\quad + \sum_k \int_0^x \int_r^\infty \exp\left[-\zeta_j(r,x)\right]\overline{f}_{jk}(r+\nu_j z, r')\Psi_k(x-z,r')dr'dz \\ \zeta_j(r,t) &= \int_0^t \sigma\left(r+\nu_j t'\right)dt' \end{aligned} \tag{5b}$$

where the exponential $\exp\left[-\zeta_j(r,x)\right]$ is an integrating factor. Wilson, Jones, Maiden, and Goldhagen (2003) point out that the total cross section $\sigma_j(E)$ with the medium for each particle type of energy E may be expanded as

$$\sigma_j(E) = \sigma_{j,at}(E) + \sigma_{j,el}(E) + \sigma_{j,r}(E) \tag{5c}$$

where the first term refers to collision with atomic electrons, the second term is for elastic nuclear scattering, and the third term describes nuclear reactions. They point out that the microscopic cross sections for most particles are ordered as given in Table [1] below.

Table [1] (From Wilson et al., 2003)

Cross Section	Cross Section Value	Average Energy Transfer
$\sigma_{j,at}(E)$	10^{-16} cm^2	ΔE_{at} 10^2 eV
$\sigma_{j,el}(E)$	10^{-19} cm^2	ΔE_{at} 10^6 eV
$\sigma_{j,r}(E)$	10^{-24} cm^2	ΔE_{at} 10^8 eV

Given the complex geometries of the shuttle craft, equation (5a) must be solved numerically.

2.0 Numerical Solution of the Forward Boltzmann Equation

2.1 Adomian's Decomposition Method. Following Adomian's construction [1,2], we re-express equation(5a) as

$$\left[L_x\Psi_j(x,r) - \nu_j \frac{\partial}{\partial r}\Psi_j(x,r) + \sigma_j(r)\right] \Psi(x,r) = \sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_k(x,r')dr', \tag{6}$$

where L_x is defined as $\frac{\partial}{\partial x}$. Multiplying of equation (6) through by L_x^{-1} we obtain

$$\begin{aligned} L_x^{-1}L_x\Psi_j(x,r) - L_x^{-1}\nu_j \frac{\partial}{\partial r}\Psi_j(x,r) + L_x^{-1}\sigma_j(r)\Psi(x,r) \\ = L_x^{-1}\sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_k(x,r')dr' \end{aligned} \tag{7}$$

Observing that $L_x^{-1} = \int_0^x (\bullet)\, dx'$, equation (7) may be re-expressed as follows

$$[\Psi_j(x,r) - \Psi_j(0,r)] - L_x^{-1}\nu_j \frac{\partial}{\partial r}\Psi_j(x,r) + L_x^{-1}\sigma_j(r)\Psi(x,r) = L_x^{-1}\sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_k(x,r')dr' \tag{8}$$

Rearranging equation (8), we may express $\Psi_j(x,r)$ as follows

$$\Psi_j(x,r) = \Psi_j(0,r) + L_x^{-1}\nu_j \frac{\partial}{\partial r}\Psi_j(x,r) - L_x^{-1}\sigma_j(r)\Psi(x,r) + L_x^{-1}\sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_k(x,r')dr' \tag{9}$$

Assume that $\Psi_j(x,r)$ is expressible in the form

$$\Psi_j(x,r) = \sum_{m=0}^{\infty} \Psi_j^{(m)}(x,r) \tag{10}$$

then we may express equation (10) as follows

$$\Psi_j(x,r) = \Psi_j(0,r) + L_x^{-1}\nu_j \frac{\partial}{\partial r}\left[\sum_{m=0}^{\infty} \Psi_j^{(m)}(x,r)\right] - L_x^{-1}\sigma_j(r)\left[\sum_{m=0}^{\infty} \Psi_j^{(m)}(x,r)\right] + L_x^{-1}\sum_k \int_r^\infty \overline{f}_{jk}(r,r')\left[\sum_{m=0}^{\infty} \Psi_j^{(m)}(x,r')\right]dr' \tag{11}$$

Assuming all of the appropriate continuity arguments, we may re-express equation (11) as follows

$$\Psi_j(x,r) = \Psi_j(0,r) + \sum_{m=0}^{\infty} \nu_j \frac{\partial}{\partial r} L_x^{-1}\Psi_j^{(m)}(x,r) - \sum_{m=0}^{\infty} \sigma_j(r) L_x^{-1}\Psi_j^{(m)}(x,r) + \sum_{m=0}^{\infty}\sum_k \int_r^\infty \overline{f}_{jk}(r,r') L_x^{-1}\Psi_j^{(m)}(x,r')dr' \tag{12}$$

Extracting the $m = 0$ term from each of the series terms in equation (12), we re-write (12) as follows

$$\begin{aligned}\Psi_j(x,r) = \Psi_j(0,r) &+ \nu_j \frac{\partial}{\partial r} L_x^{-1}\Psi_j^{(0)}(x,r) + \sum_{m=1}^{\infty} \nu_j \frac{\partial}{\partial r} L_x^{-1}\Psi_j^{(m)}(x,r) \\ &- \sigma_j(r) L_x^{-1}\Psi_j^{(0)}(x,r) - \sum_{m=1}^{\infty} \sigma_j(r) L_x^{-1}\Psi_j^{(m)}(x,r) \\ &+ \sum_k \int_r^\infty \overline{f}_{jk}(r,r') L_x^{-1}\Psi_j^{(0)}(x,r')dr' \\ &+ \sum_{m=1}^{\infty}\sum_k \int_r^\infty \overline{f}_{jk}(r,r') L_x^{-1}\Psi_j^{(m)}(x,r')dr' \end{aligned} \tag{13}$$

We already know that $\Psi_j^{(0)}(x,r)$ is a prespecified function. In point of fact, it is exactly $\Psi_j(0,r)$ for all values of $j = 1, 2, \ldots,$. We also know that, by definition, $\Psi_j(0,r)$ is not a function of x. Hence, we may express $L_x^{-1}\Psi_j^{(0)}(x,r)$ as follows

$$\Psi_j^{(0)}(x,r) = L_x^{-1}\Psi_j(0,r) = \int_0^x \Psi_j(0,r)d\zeta = \Psi_j(0,r)\int_0^x d\zeta = \Psi_j(0,r)x \qquad (14)$$

Substituting equation (14) into equation (13) yields

$$\begin{aligned}\Psi_j(x,r) = \Psi_j(0,r) &+ \nu_j \frac{\partial}{\partial r}\left[\Psi_j(0,r)\right]x + \sum_{m=1}^{\infty}\nu_j\frac{\partial}{\partial r}L_x^{-1}\left[\Psi_j^{(m)}(x,r)\right] \\ &- \sigma_j(r)L_x^{-1}\left[\Psi_j(0,r)\right]x - \sum_{m=1}^{\infty}\sigma_j(r)L_x^{-1}\left[\Psi_j^{(m)}(x,r)\right] \\ &+ \sum_k\int_r^{\infty}\overline{f}_{jk}(r,r')\Psi_j(0,r')x dr' \\ &+ \sum_{m=1}^{\infty}\sum_k\int_r^{\infty}\overline{f}_{jk}(r,r')L_x^{-1}\left[\Psi_j^{(m)}(x,r')\right]dr' \qquad (15)\end{aligned}$$

Grouping all of the x terms together, we may express $\Psi_j(x,r)$ as follows

$$\begin{aligned}\Psi_j(x,r) = \Psi_j(0,r) + x&\left[\nu_j\frac{\partial}{\partial r}\Psi_j(0,r) - \sigma_j(r)\Psi_j(0,r) + \sum_k\int_r^{\infty}\overline{f}_{jk}(r,r')\Psi_j(0,r')dr'\right] \\ &+ \sum_{m=1}^{\infty}\nu_j\frac{\partial}{\partial r}L_x^{-1}\Psi_j^{(m)}(x,r) \\ &- \sum_{m=1}^{\infty}\sigma_j(r)L_x^{-1}\Psi_j^{(m)}(x,r) \\ &+ \sum_{m=1}^{\infty}\sum_k\int_r^{\infty}\overline{f}_{jk}(r,r')L_x^{-1}\Psi_j^{(m)}(x,r')dr' \qquad (16)\end{aligned}$$

From equation (16), we observe that

$$\begin{aligned}&\Psi^{(0)}(x,r) = \Psi_j(0,r) \\ &\Psi^{(1)}(x,r) = \Psi_j(0,r) + x\left[\nu_j\frac{\partial}{\partial r}\Psi_j(0,r) - \sigma_j(r)L_x^{-1}\Psi_j(0,r) + \sum_k\int_r^{\infty}\overline{f}_{jk}(r,r')\Psi_j(0,r')dr'\right] \qquad (17)\end{aligned}$$

From which we may construct the following recursion relationship

$$\begin{aligned}\Psi_j^{(m+1)}(x,r) = \nu_j\frac{\partial}{\partial r}&\left[L_x^{-1}\Psi_j^{(m)}(x,r)\right] - \sigma_j(r)L_x^{-1}\Psi_j^{(m)}(x,r) \\ &+ \sum_k\int_r^{\infty}\overline{f}_{jk}(r,r')\Psi_j^{(m)}(x,r')dr' \qquad (18)\end{aligned}$$

Making use of equation (18) and our results in equation (17), we construct $\Psi_j^{(2)}(x,r)$

$$\begin{aligned}\Psi_j^{(2)}(x,r) = {} & \nu_j \frac{\partial}{\partial r} L_x^{-1} \left[\nu_j \frac{\partial}{\partial r} \Psi_j(0,r) - \sigma_j(r)\Psi_j(0,r) + \sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_j(0,r')dr' \right] \frac{x}{1!} \\ & - \sigma_j(r) L_x^{-1} \left[\nu_j \frac{\partial}{\partial r} \Psi_j(0,r) - \sigma_j(r)\Psi_j(0,r) + \sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_j(0,r')dr' \right] \frac{x}{1!} \\ & + \sum_k \int_r^\infty \overline{f}_{jk}(r,r') L_x^{-1} \left[\nu_j \frac{\partial}{\partial r} \Psi_j(0,r) - \sigma_j(r)\Psi_j(0,r) + \sum_k \int_r^\infty \overline{f}_{jk}(r,r'')\Psi_j(0,r'')dr'' \right] \frac{x}{1!}\end{aligned} \tag{19}$$

Performing the necessary integrations in equation (19), we see that $\Psi_j^{(2)}(x,r)$ becomes

$$\begin{aligned}\Psi_j^{(2)}(x,r) = {} & \nu_j \frac{\partial}{\partial r} \left[\nu_j \frac{\partial}{\partial r} \Psi_j(0,r) - \sigma_j(r)\Psi_j(0,r) + \sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_j(0,r')dr' \right] \frac{x^2}{2!} \\ & - \sigma_j(r) \left[\nu_j \frac{\partial}{\partial r} \Psi_j(0,r) - \sigma_j(r)\Psi_j(0,r) + \sum_k \int_r^\infty \overline{f}_{jk}(r,r')\Psi_j(0,r')dr' \right] \frac{x^2}{2!} \\ & + \sum_k \int_r^\infty \overline{f}_{jk}(r,r') \left[\nu_j \frac{\partial}{\partial r} \Psi_j(0,r) - \sigma_j(r)\Psi_j(0,r) + \sum_k \int_r^\infty \overline{f}_{jk}(r,r'')\Psi_j(0,r'')dr'' \right] \frac{x^2}{2!}\end{aligned} \tag{20}$$

Higher order terms $\Psi_j^{(m)}(x,r) \quad m \geq 3$ may be recursively generated from the lower order terms presented.

Acknowledgements. I would like to thank John W. Wilson, NASA Langley, for his constant support of this research effort, for supplying many of the needed references, and for his discussions and reading of earlier versions of this manuscript. Additionally, I would like to thank George Adomian, GAC Corporation and developer of the decomposition method, for his discussions and support, as well as for his discussions on how to apply his method to solving partial differential equation models.

References

1. G. Adomian, *A review of the decomposition method and some recent results for nonlinear equations*, Mathl. Comput. Modeling, **13#7** (1990) 17–43.
2. G. Adomian, **Solving Frontier Problems of Physics – The Decomposition Method**, (Kluwer Acad. Publ., N.Y., 1994).
3. M.S. Clowdsley, J.H. Hienbockel, F.F. Badavi, and J.W. Wilson, *Shuttle induced neutron environment: Computational requirements and validation*, NASA Technical Paper 2002-02-2460 (NASA Langley Research Center, 2002).
4. F.A. Cucinotta, *Calculations of cosmic-ray helium transport in shielding materials*, NASA Technical Paper 3354 (NASA Langley Research Center, 1993).

5. F.A. Cucinotta, F. Hajnal, and J.W. Wilson, *Energy deposition at the bone-tissue interface from target fragments produced by high energy nucleons*, Health Phys., **59** (1990) 819–825.
6. F. Cucinotta, R. Katz, J.W. Wilson, L.W. Townsend, J. Shinn, and F. Hajnal, *Biological effectiveness of high-energy protons: Target fragmentation*, Radiation Research, **127** (1991) 130–137.
7. F.A. Cucinotta, L.W. Townsend, and J.W. Wilson, *Description of alpha-nucleus interaction cross sections for cosmic ray shielding studies*, NASA Technical Paper 3285 (NASA Langley Research Center, 1993).
8. F. Cucinotta, L.W. Townsend, J.W. Wilson, M.J. Golightly, and M. Weyland, *Analysis of radiation risk from alpha particle component of solar particle events*, Adv. Space Res., **14#10** (1994) 661–670.
9. F.A. Cucinotta, J.W. Wilson, and F. F. Badavi, *Extension of the BRYNTRN code to monoenergetic light ion beams*, NASA Technical Paper 3472 (NASA Langley Research Center, 1994).
10. J.W. Wilson, *Heavy ion transport in the straight ahead approximation*, NASA TP-2178 (NASA Langley Research Center, 1983).
11. J.W. Wilson, L.W. Townsend, and F. Kahn, *Evaluation of highly ionizing components in high-energy nucleon radiation fields*, Health Phys., **57** (1989) 717–724.
12. J.W. Wilson, L.W. Townsend, W. Schimmerling, G.S. Khandelwal, F. Khan, J.E. Nealy, F.A. Cucinotta, L.C. Simonsen, J.L. Shinn, and J.W. Norbury, *Transport methods and interactions for space radiations*, NASA RP-1257 (NASA Langley Research Center, 1991).
13. J.W. Wilson, F.A. Cucinotta, J.L. Shinn, and L.W. Townsend, *Target fragmentation in radiobiology*, NASA Technical Memorandum 4408 (NASA Langley Research Center, 1993).
14. J. W. Wilson, L.W. Townsend, J.E. Healy, S.Y. Chun, B.S., Hong W.W. Buck, S.L. Lamkin, B.D. Ganapol, F. Khan, and F.A. Cucinotta, *BRYNTRN: A Paryon Transport Model*, NASA TP-2887 (NASA Langley Research Center, 1989).
15. J. W. Wilson, I.W. Jones, D.L. Maiden, and P. Goldhagen, *Atmospheric ionizing radiation (AIR): Analysis, results, and lessons learned from the June 1997 ER-2 campaign*, NASA/CP-2003-212155 (NASA Langley Research Center, 1997).